3-Dimensional Geometry: Assessment of Students' Responses

Christine Lawrie
University of New England
<clawri2@metz.une.edu.au>

John Pegg
University of New England
<jpegg@metz.une.edu.au>

The authors formulated three research questions in association with the development of a test in 3-dimensional geometry in response to Mariotti's study on the complexity in mental imaging using nets of solids. This paper reports on the coding of responses to a question on nets, using two frameworks, the SOLO Taxonomy and the van Hiele levels of understanding. The findings show that students' descriptions of nets can be coded in both frameworks, thus supporting the three research questions.

Introduction

An important part of growth in insight for 3-dimensional geometry is visual imaging. Through imaging, the development of understanding of the relationships within a figure is promoted. Researchers continue to be interested in understanding the nature of growth in students' insight into 3-dimensional geometry, and in the role played by visual-based processes (Despina, Leikin & Silver, 1999; Gutiérrez, 1996; Mariotti, 1989; Mitchelmore 1980). Gutiérrez (1996, p.1-9) defines visualisation in mathematics as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, and that it contains four main elements, mental images, external representations, processes, and abilities of visualisation. He considers that visualisation is one of the main bases of cognition (p.1-3), while Mitchelmore (1980, p.83) argues that "it is of great value to be able to visualize and represent three-dimensional configurations and to comprehend the geometrical relations among the various parts of a figure." Students' ability to decompose and recombine images, that is the ability to break down a visual image into simpler parts and them recombine those parts into new images, is an important component of imagery (Brown & Wheatley, 1997

Two ways in which the ability to translate between 3-dimensional solids and their 2-dimensional representations can be demonstrated are, first, through the 2-dimensional geometrical drawing of 3-dimensional solids, and, second, through the development of 2-dimensional nets of 3-dimensional solids. A net has been described by Borowski and Borwein (1991) as a diagram of a hollow solid consisting of the plane shapes of the faces so arranged that the cut-out diagram could be folded to form the solid. The first method of representation, the 2-dimensional geometric drawing, was explored by Mitchelmore (1980), and the second, students' ability to recognise and design nets, was investigated by Mariotti (1989) and Despina, Leikin and Silver (1999).

Mariotti (1989), in her attempt to identify specific didactic variables related to the utilization of mental images, hypothesised that there are two levels of complexity in the manipulation of mental images:

> a first level when primary intuitions are sufficient: the image is global, it is not necessary to coordinate intermediate processes to solve the problem:
> a second level when the primary intuitions are not sufficient any more: an operative organization of images is required to coordinate them according to the composition transformation. (p.260)

In her study, she found that "constructing the correct net of the solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (faces, vertices and edges)" (p. 263), and that it is possible to construct a hierarchy of difficulties in relation to the elaboration of mental images (p. 265). Despina, Leikin and Silver (1999, pp. 4-248), in their examination of the problems related to finding different nets of a cube, found that students experienced great difficulty when attempting to work systematically.

While Mariotti (1989, p. 263) argued that "constructing the correct net of a solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (faces, vertices and edges)", Despina, Leikin and Silver (1999, pp. 4-242), in investigating students' ability to transform the cube to different nets, found that "there is little information with respect to how students coordinate and analyze the components of a 3D solid when transforming it into a 2D net or vice versa."

Research Design

The authors formulated three research questions related to Mariotti's study:

1. Can Mariotti's two levels of difficulty be demonstrated in students' descriptions of nets?
2. Do Mariotti's levels correspond with recognised frameworks of insight?
3. Is there a hierarchy of difficulties within Mariotti's second level?

To investigate these research questions and others, the authors developed a test, the responses to which could be analysed for information about students' understanding of 3-dimensional geometry. The questions covered the solid form and its cross-sections, as well as nets of 3-dimensional figures. This report considers the coding of the students' responses to one of the questions in which students expressed their understanding of a net. Two frameworks, the SOLO Taxonomy (Biggs & Collis, 1982) and the van Hiele levels of understanding (van Hiele, 1986) are used in the coding of the nature of thinking displayed by the students in their responses.

Background

The SOLO Taxonomy (Biggs & Collis, 1982) has been identified by several writers (e.g., Pegg & Davey, 1998) as having strong similarities with the van Hiele Theory, despite having some philosophical differences. It is concerned with evaluating the quality of students' responses to various items. A SOLO classification involves two aspects. The first of these is a mode of functioning, and, the second, a level of quality of response within the targeted mode. Of relevance to this study are the three modes, ikonic, concrete symbolic, and formal. Within each of these modes, students may demonstrate a unistructural, multistructural or relational level of response.

In the *ikonic mode* students internalise outcomes in the form of images and can be said to have intuitive knowledge. By contrast, in the *concrete symbolic mode,* students are able to use or learn to use a symbol system such as a written language and number notation. This is the most common mode addressed in learning in the upper primary and secondary schools. When operating in the *formal mode*, the student is able to consider more abstract concepts, and to work in terms of 'theories'. A description of the levels that occur within the modes is given:

The *unistructural level* of response draws on only one concept or aspect from all those available.

The *multistructural level* of response is one that contains several relevant but independent concepts or aspects.

The *relational level* of response is one that relates concepts or aspects. The relevant concepts are woven together to form a coherent structure.

The targeting of the concrete symbolic mode for instruction in primary and secondary schools, and the implication that most students are capable of operating within the concrete symbolic mode (Collis, 1988) has resulted in the exploration of the nature of student responses within the mode. This has led to the identification of at least two unistructural (U) - multistructural (M) - relational ® cycles within the concrete symbolic mode (Pegg & Davey 1998). The most noticeable characteristic of the cyclic form of the levels is that the relational response (R1) in the first cycle becomes the unistructural element (U2) in the second cycle. This cyclical nomenclature has been used in the coding of concrete symbolic responses for this study.

Pierre van Hiele's (1986) work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, the ones commonly displayed by secondary students and most relevant to this study, is given for 3-dimensional geometry (Pegg's (1997) differentiation between Levels 2A and 2B is used):

Level 1. Perception is visual only. A solid is seen as a total entity and as a specific shape. Students are able to recognise solids and to distinguish between different solids. Properties play no explicit part in the recognition of the shape, even when referring to faces, edges or vertices.

Level 2A. A solid is identified now by a single geometric property rather than by its overall shape. For example, a cube may be recognised by either its twelve equal edges or its six square faces.

Level 2B. A solid is identified in terms of its individual elements or properties. These are seen as independent of one another. They include side length, angle size, and parallelism of faces.

Level 3. The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised. Symmetry follows as a consequence. Simple proofs and informal deductions are justified. Families of solids can be classified.

Level 4. Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions in definitions are used with understanding, as are equivalent definitions for the same concept.

Van Hiele saw his levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s). He also saw (i) the levels as discontinuous;that is, students do not move through the levels smoothly, (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a new level, and (iii) students at different levels speaking a 'different language' and having a different mental organisation.

Design and Initial Analysis of Responses

Among the questions in the 3-dimensional geometry test designed by the authors were six questions investigating students' understanding of nets. These questions probed for identification, description, explanation of relations and construction of nets. Initially, the test was given to over 1000 students from all except the final year (i.e., from Years 7 to 11) of four secondary schools in a rural city in New South Wales, Australia. This paper considers the responses to one of the questions on nets, namely:

Describe in as much detail as possible what is a net of a solid.

Responses to the question were grouped according to the depth displayed in each answer. In an attempt to find some broad descriptors for the coding, the responses were initially grouped into three sections. Descriptions of each group, together with examples of responses matching the descriptors are given below.

Group A Responses

Responses in this group identified students who considered the net solely as a global image. There was no indication of awareness of a process of transformation between the solid and the net, or of properties or parts of the solid.

- what the solid is before it is a solid (A1)
- the 2D thing of a solid (A2)
- a solid flat (A3)
- a flat drawing of a solid (A4)
- is drawn like below (diagram) (A5)
- the outside of a solid (A6)

Group B Responses

In these responses, students showed awareness of the process of conversion from the 2-D to the 3-D image and/or vice versa, by folding/unfolding, putting together/taking apart, or by spreading out flat. However, there was no indication of awareness of the properties or parts of either the solid shape or the net.

- the unfolded 3 dimensional shape (B1)
- something that can fold in to make an object; for example, this is a pyramid net of a solid (diagram) (B2)
- the solid taken apart and flat (B3)
- a shape spread out ... when you fold it back together it forms that shape (B4)
- the combination of 2D shapes that you need to form a 3D shape (B5)
- the pattern made before it is put together ... the outline of the solid (B6)
- the shape and parts of an enclosed space, when it is taken apart and laid flat to make a shape (B7)
- a plan of a solid that has been opened (B8)
- what I have drawn on the right (diagram) ... this net makes a triangular prism (B9)

Group C Responses

Responses in this group contrasted to those in group B in that in addition to the process of transformation between a solid and a net, the students showed in their answer that they were aware of at least one part (faces, edges and vertices) of a solid. Included in these rcsponses were those which indicated the need for edges to correspond.

- a description of the solid with its faces, edges, and vertices in a flat layered out position ... it shows the different faces on sections of the shape (C1)
- the shape and all its faces folded out (C2)
- a solid kind of flattened out ... a plane shape with sides on it (C3)
- the plan (like a building plan) that shows all the sides in one picture (C4)
- a 3-dimensional shape flattened out to make a 2-dimensional shape ... the edges are separated to make it flat (C5)
- the faces laid out flat ... you can see only the framework for a shape (C6)
- the outline of the skeleton of a solid ... it is a 2D depiction of the solid as if it were

unfolded ... from the net, each line represents an edge and each area represents a face ... when make into a 3D solid, the vertices form as a result of joining the edges and faces together (C7)

- a 2-D representation of a solid ... when all sides are joined together it must make a solid, so these 2-D shapes must be in the correct position on the net ... the sides of the 2-D net must be correct lengths so that when they are joined together there must be no uneven or left over edges (C8)

It is considered that the above grouping supports the first research question; that is, it demonstrates Mariotti's two levels of difficulty. Group A corresponds with the first level in that the responses indicate solely a global perception of a net, whereas, groups B and C, in demonstrating awareness of the process of transformation between the 3-dimensional and 2-dimensional shapes, correspond with her second level.

Final Coding

The responses within the above initial groups were coded with relation to the SOLO Taxonomy. The first consideration was to classify the responses according to their targeted mode of functioning. Group A responses were considered to illustrate students who were perceiving a net intuitively as a visual image. Hence, these responses were classified as being of the ikonic mode. The responses in the latter two groups, B and C, were both considered to be perceiving the net as a symbol or representation of the solid, with group C perceiving the net additionally as corresponding to the parts of the solid. Responses belonging to both these groups were classified as being of the concrete symbolic mode. No responses were considered to display the formal mode of functioning.

The nature of the responses for group A was not considered further. However, groups B and C responses, both deemed as concrete symbolic in nature, were further classified for the cyclical levels within the mode. It was considered that responses in group B, those displaying awareness of transformation between the 3-dimensional solids and 2-dimensional nets, but not identifying any correspondence between the parts of the solid and the shape of the net, were in the first cycle. These responses were coded for the level demonstrated, according to whether they indicated a single process in the transformation; for example, unfolded/flattened, or whether they considered the transformation as a multiple process;that is, those responses which indicated either that the process was reversible, or that it consisted of two or more actions; for example, cut and unfolded. Responses of the first type were coded as being unistructural in the first cycle (U1). In the examples, it is considered the responses B1, B5, B8, and B9 are unistructural, first cycle. Those of the second type were coded as being multistructural in the first cycle (M1). Responses coded as M2 from the examples are B2, B3, B4, B6, and B7. R1, or relational responses of the first cycle become the unistructural element (U2) in the second cycle.

Responses which identified part(s) or property(ies) of a solid were deemed to belong to the second cycle of levels. Identification of a single part or property indicated a response as being unistructural (U2), for example, responses C2, C3, C4, C5, and C6. Reference to more than one property or part of the solid classified a response as multistructural (M2), for example, response C1. If a response also included information concerning a correspondence between the faces or edges of the solid and the sections of the net, or of the matching of edges in the reconstruction of the solid from the net, the response was coded as relational (R2). Responses which matched this coding are C7 and C8.

Finally, the codings were related to the van Hiele descriptors. It was considered that the ikonic responses indicated a visual understanding, corresponding to van Hiele Level 1. Responses which demonstrated an appreciation of the physical process involved in the transformation between a solid and its net only were considered not to show awareness of properties, yet were exhibiting more than visual awareness. Hence, these responses have been coded as transition; that is, between van Hiele Levels 1 and 2A. Responses which indicated awareness of a single property or part of a solid, and were coded as U2 were considered to demonstrate van Hiele Level 2A understanding, while responses which were similar but showed also awareness of more than one property or part of a solid were considered to demonstrate van Hiele Level 2B understanding. Finally, it was considered that responses which included a reference to the relationship between the faces and/or edges of the solid and the net demonstrated understanding at van Hiele Level 3.

A summary of the coding of the nature of thinking displayed in responses to the question 'Describe in as much detail as possible what is a net of a solid' is given in Table 1.

Table 1
Coding of the Nature of Thinking Displayed in Responses on Nets of Solids

Description	SOLO mode/level	van Hiele Level
Visual only		
a flattened shape	Ikonic	1
Perceiving the process		
a shape folded/unfolded	CS/U1	transition
the above plus a linkage (cut, take apart, make up)	CS/M1	transition
identifying part or parts of the net	CS/R1	transition
Considering the shapes		
edges/faces matching or some property of the net	CS/U2	2A
combination of the above	CS/M2	2B
relational position of the faces	CS/R2	3

The coding of the responses using the two frameworks, the SOLO Taxonomy and the van Hiele levels of understanding, not only confirms the first research question, that Mariotti's two levels of difficulty can be identified in students' descriptions of nets, but also supports the second and third research questions. Mariotti's first and second levels correspond respectively with the ikonic and concrete symbolic modes in the SOLO Taxonomy, and also with van Hiele Level 1, and van Hiele Levels 2 and 3. Further, the coding within the two frameworks clearly demonstrates that Mariotti's second level can be partitioned into a hierarchy of degrees of insight. This is shown in the two cycles of levels in the SOLO Taxonomy and in the correspondence between Mariotti's second level and van Hiele's Levels 2 and 3.

Conclusion

In this study, the authors have investigated further the insight required in visual reasoning in relation to nets of 3-dimensional solids. Three research questions developed from Mariotti's study have been demonstrated to hold true. This now provides the theoretical evidence necessary to explore the issues further.

References

Biggs, J. & Collis, K. (1982). Evaluating the quality of learning (Structure of the observed learning outcome). New York: Academic Press.

Borowski, E. J. & Borwein, M. M. (1991). The Harper Collins dictionary of mathematics. New York: Harper Perennial.

Brown, D. L. & Wheatley, G. (1997). Components of imagery and mathematical understanding, Focus on Learning Problems in Mathematics, 19, 45-70.

Collis, K. (1988). The add up or take syndrome. In J. Pegg (Ed.), Mathematical interfaces (pp. 66-75). Adelaide: Australian Association of Mathematics Teachers

Despina, A. S., Leikin, R. & Silver, E. A. (1999). Exploring students' solution strategies in solving a spatial visualization problem involving nets. In O. Zaslavsky (Ed.), Proceedings of the 23rd annual conference of The International Group for the Psychology of Mathematics Education 4, 241-248. Haifa (Israel): Program Committee.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: in search of a framework. In L. Puig & A. Gutiérrez (Eds.), Proceedings of the 20th annual conference of The International Group for the Psychology of Mathematics Education 1, 3-19. Valencia (Spain): Program Committee.

Mariotti, M. A. (1989). Mental images: some problems related to the development of solids. Paper presented at the 13th annual conference of The International Group for the Psychology of Mathematics Education 2, 258-265. Paris (France).

Mitchelmore, M. C. (1980). Prediction of developmental stages in the representation of regular space figures. Journal for Research in Mathematics Education, 11, 83-93

Pegg, J. (1997). Broadening the descriptors of van Hiele's levels 2 and 3. In F. Biddulph & K. Carr (Eds.), People in mathematics education. Proceedings of the 20th annual conference of the Mathematics Education Research Group of Australasia, 391-396. Rotorua: MERGA.

Pegg, J. & Davey, G. 1998. Interpreting student understanding in geometry: A synthesis of two models. In R. Lehrer & D. Chazan (Eds.), Designing learning environments for developing understanding of geometry and space, (109-135). Mahwah, New Jersey: Lawrence Erlbaum Ass.

van Hiele, P. M. (1986) Structure and insight. A theory of mathematics education. Florida: Academic Press.